

# Modified gravity: $F(R) = R \exp(\alpha R)$

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## Abstract

A novel modified theory of gravity with the function  $F(R) = R \exp(\alpha R)$  instead of Ricci scalar  $R$  in the Einstein–Hilbert action is suggested and analyzed. The action is converted into Einstein–Hilbert action at small value of the parameter  $\alpha$ . From local tests we obtain a bound on the parameter  $\alpha \leq 10^{-6} \text{ cm}^2$ . The Jordan and Einstein frames are considered and the potential of the scalar field in Einstein’s frame is found. The static solutions of the model are obtained corresponding to the Schwarzschild–de Sitter space. We show that the de Sitter space is unstable but a solution with zero curvature is stable. It was demonstrated that the model passes the matter stability test.

## 1 Introduction

Astronomical data indicate that the Universe accelerates at the present time. The nature of the driving force that results the accelerated expansion is unknown yet. There are several approaches to explain cosmic acceleration. The interpretation of currently accelerating Universe, within the General Relativity (GR), requires the introduction of dark energy - exotic substance with large negative pressure  $P_{DE}$  so that  $P_{DE} \simeq -\rho_{DE}$  ( $\rho_{DE}$  is the dark energy density). The similar scheme uses the cosmological constant  $\Lambda$  [1]. Such a model gives a good description of all observational data. However, with theoretical point of view, it is not clear how to explain the introduction of a new physical constant  $\Lambda$  which is very small compared with vacuum energy of elementary particles. Models with dynamical dark energy include a new scalar field [2]. Another way to describe the acceleration of the early and late Universe is the modification of GR. So-called  $F(R)$ -gravity theories replace the Ricci scalar in Einstein–Hilbert action by the function  $F(R)$  [3], [4], [5]. Such purely gravitational models present an alternative to  $\Lambda$ CDM ( $\Lambda$ -Cold Dark Matter) model and may clear up the coincidence problem, and

describe the inflation and late-time acceleration. In  $F(R)$ -gravity models the cosmic acceleration is due to new gravitational physics.

In this paper a novel model of exponential-like  $F(R)$ -gravity is suggested. The paper is organized as follows. In Sec.2, we consider a model of modified gravity with the exponential-like Lagrangian density. A bound on the parameter  $\alpha$  with the dimension  $(\text{length})^2$  is obtained. We find static Schwarzschild–de Sitter solutions in Jordan’s frame and describe FRW (Friedmann –Robertson–Walker) cosmology in Sec.3. The potential of the scalar field in the scalar-tensor form of the model (Einstein’s frame) is obtained in Sec.4. It is shown that the de Sitter space is unstable and the Minkowski space corresponding to a solution with zero Ricci scalar is stable. In Sec.5 the matter stability of the model is investigated and we demonstrate that the model passes the matter stability test at  $R_0 < 2/\alpha$ . We discuss results obtained in Sec.6.

The Minkowski metric  $\eta_{\mu\nu}=\text{diag}(-1, 1, 1, 1)$  is used and  $c = \hbar = 1$  is assumed throughout the paper.

## 2 The modified gravity model

We suggest the modified gravitational theory with the function  $F(R)$  instead of the Ricci curvature  $R$  ( $R \rightarrow F(R)$ ) in the Einstein–Hilbert action:

$$F(R) = R \exp(\alpha R), \quad (1)$$

so that the action is

$$S = \int d^4x \sqrt{-g} \mathcal{L} = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} F(R) + \mathcal{L}_m \right], \quad (2)$$

where  $\kappa = \sqrt{8\pi m_{Pl}^{-1}}$ ,  $g=\det g_{\alpha\beta}$  ( $g_{\alpha\beta}$  is a metric tensor),  $m_{Pl} = G^{-1}$  is the Planck mass,  $G$  is the gravitation (Newton) constant,  $\mathcal{L}_m$  is the matter Lagrangian density. The action (2) is written in the Jordan frame. It should be mentioned that other variants of exponential gravity were considered in [6], [7]. Thus, the constant  $\alpha$  with the dimension of  $(\text{length})^2$  is introduced. To pass the Solar System tests the constant  $\alpha$  should be small compared with  $R^{-1}$  ( $\alpha R \ll 1$ ) because the deviation from GR based on the Einstein–Hilbert action has to be diminutive. As a result one can obtain from (1) the Taylor series

$$F(R) = R + \alpha R^2 + \frac{1}{2}\alpha^2 R^3 + .... \quad (3)$$

Because  $\lim_{\alpha \rightarrow 0} F(R) = R$  action (2) of our model approaches to the Einstein–Hilbert action at  $\alpha R \ll 1$ . GR passes local tests and, therefore, one may obtain a restriction on the parameter  $\alpha$  from observational data. From the Eöt-Wash experiment [8], [9] (see also [10], [11]), we obtain a laboratory bound on the parameter  $\alpha$ :

$$\alpha \leq 10^{-6} cm^2. \quad (4)$$

It should be mentioned that F(R)-gravity with the function  $F(R) = R + R^2/6M^2$  ( $M$  has a dimension of the mass) was considered in [12] which is the approximation to series (3) at small  $\alpha R$  (the value  $\alpha R$  is dimensionless). The modified  $R^2$ -gravity is insensitively investigated [13], [14] because quantum corrections to GR include  $R^2$  counter term [15], [16], [17] and such a model possesses attractive features: an absence singularity in the past and in the future and gives the self-consistent inflation, etc. Let us consider the matter Lagrangian density  $\mathcal{L}_m$ , entering (2), which represents the perfect fluid with the energy-momentum tensor

$$T_{\alpha\beta}^{mat} = (P^{mat} + \rho^{mat}) u_\alpha u_\beta + P^{mat} g_{\alpha\beta}, \quad (5)$$

where  $P^{mat}$  is a pressure,  $\rho^{mat}$  is the energy density, and the four-velocity of the fluid obeys  $u^\alpha u_\alpha = -1$ . Then equations of motion following from Eq.(2) are given by

$$R_{\mu\nu} F'(R) - \frac{1}{2} g_{\mu\nu} F(R) + g_{\mu\nu} g^{\alpha\beta} \nabla_\alpha \nabla_\beta F'(R) - \nabla_\mu \nabla_\nu F'(R) = \kappa^2 T_{\mu\nu}^{mat}, \quad (6)$$

where a covariant derivative is  $\nabla_\mu$ ,  $F'(R) = dF(R)/dR$ . For FRW metric, the conservation of the energy-momentum tensor  $\nabla^\mu T_{\mu\nu}^{mat} = 0$  results:

$$\dot{\rho}^{mat} + 3H (\rho^{mat} + P^{mat}) = 0. \quad (7)$$

Here the Hubble parameter is  $H = \dot{a}(t)/a(t)$ , where  $a(t)$  is a scale factor and a over dot denotes the differentiation with respect to the time. It follows from Eq.(7) that for the fluid with the property of the dark energy when the equation of state (EoS) is  $P^{mat} = -\rho^{mat}$ , the energy density  $\rho^{mat}$  is a constant.

### 3 Static Solutions

Let us consider solutions to Eq. (6) for action (2) in a case with a constant Ricci scalar  $R = R_0$ . For EoS:  $P^{mat} = -\rho^{mat}$  ( $\rho^{mat}$  is a constant) Eq. (6) reads [4]

$$2F(R_0) - R_0 F'(R_0) = 4\kappa^2 \rho^{mat}. \quad (8)$$

From Eq.(8), we obtain the equation as follows:

$$R_0 (1 - \alpha R_0) = 4\kappa^2 \rho^{mat} \exp(-\alpha R_0). \quad (9)$$

Because transcendent equation (9) can be solved only by the numerical computation, we restrict our consideration by the case of free space ( $\rho^{mat} = 0$ ). Then there are two solutions to Eq. (9) at  $\rho^{mat} = 0$ :

$$R_0 = 0, \quad R_0 = \frac{1}{\alpha}. \quad (10)$$

The conditions of classical and quantum stability [13]  $F'(R) > 0$ ,  $F''(R) > 0$  are realized in our model for  $\alpha > 0$ ,  $R > 0$ . As a result, both solutions lead to the Schwarzschild–de Sitter space. Non-trivial solution (10)  $R_0 = 1/\alpha$  corresponds to early-time inflation. The solution with vanishing curvature  $R_0 = 0$  leads to Minkowski space. Of course the concrete scenario of the Universe evolution can be described after finding the exact solutions to Eq.(6) for Ricci scalar depending on time. The Schwarzschild spherically symmetric metric is given by

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (11)$$

For the constant Ricci scalar  $R_0$ ,  $F(R)$ -gravity theories possess Schwarzschild–(anti-)de Sitter solutions [4], [5] with the function  $B(r)$ :

$$B(r) = 1 - \frac{2MG}{r} - \frac{R_0}{12} r^2, \quad (12)$$

with the mass of the black hole  $M$ . For  $R_0 > 0$  the de Sitter space is realized, and the case  $R_0 < 0$  corresponds to the anti-de Sitter space. As in our model the non-trivial solution (10) is  $R_0 = 1/\alpha > 0$ , it corresponds to the de Sitter space and the function (12) is

$$B(r) = 1 - \frac{2MG}{r} - \frac{1}{12\alpha} r^2. \quad (13)$$

Because  $\alpha > 0$  we have the classical stability of Schwarzschild black holes. Solutions of Einstein's equation with cosmological constant  $\Lambda$  have also the function of the form (12) with  $R_0 = 4\Lambda$ . Therefore, the model under consideration leads to the dynamical cosmological constant  $\Lambda = 1/(4\alpha)$  at the time when  $R = R_0 = 1/\alpha$ . Thus, even for the space without any matter, the model mimics the dark energy (the cosmological constant). The similar property of other  $F(R)$  models was discussed in [18], [19], [4], [5].

In  $F(R)$ -gravity the entropy  $S$  is given as follows [20], [21], [22], [4], [5]:

$$S = \frac{F'(R)A}{4G}, \quad (14)$$

with the area of the horizon  $A$ . Eq.(14) is the generalization of the Bekenstein-Hawking formula [23], [24] on the case of  $F(R)$ -gravity. One obtains from equation (1)  $F'(R) = (1 + \alpha R) \exp(\alpha R)$ , and entropy (14) becomes

$$S = \frac{(1 + \alpha R) \exp(\alpha R) A}{4G}. \quad (15)$$

As a result, one can introduce the effective gravitational coupling  $G_{eff} = G/(1 + \alpha R) \exp(\alpha R)$ . The nontrivial solution (10)  $R_0 = 1/\alpha$  gives the effective gravitational constant  $G_{eff} = G/(2e)$  at the time of inflation when  $R = R_0 = 1/\alpha$ .

### 3.1 FRW cosmology

The homogeneous, isotropic and spatially flat FRW cosmology is described by the space-time metric

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2). \quad (16)$$

In this case the Ricci scalar  $R$  is given by  $R = 12H^2 + 6\dot{H}$ . For static solutions  $H_0 = \text{const}$  and  $\dot{H}_0 = 0$ . Then  $H_0 = \sqrt{R_0/12}$  and we obtain from Eq.(10) for a de Sitter phase  $H_0 = 1/\sqrt{12\alpha}$ , and a scale factor becomes

$$a(t) = a_0 \exp\left(\frac{t}{2\sqrt{3\alpha}}\right), \quad (17)$$

where  $a(0)$  is a scale factor at a cosmic time  $t = 0$ . Solution (17) describes the inflation phase.

## 4 The Scalar-Tensor Form

We have formulated modified  $F(R)$ -gravity in the Jordan frame with tensor variables  $g_{\mu\nu}$ . Another description, in the Einstein frame, corresponds to the scalar-tensor theory of gravity [4], [5] with conformally transformed metric [25]

$$\tilde{g}_{\mu\nu} = F'(R)g_{\mu\nu} = (1 + \alpha R) \exp(\alpha R) g_{\mu\nu}. \quad (18)$$

In new variables equation (2) for  $\mathcal{L}_m = 0$ , in the Einstein frame, is given by

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} \tilde{R} - \frac{1}{2} \tilde{g}^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi) \right], \quad (19)$$

where  $\tilde{R}$  is defined by new metric (18). The scalar field  $\varphi$  and the potential  $V(\varphi)$  are given by equations

$$\varphi = \frac{\sqrt{3} \ln F'(R)}{\sqrt{2\kappa}} = \frac{\sqrt{3}}{\sqrt{2\kappa}} [\ln(1 + \alpha R) + \alpha R], \quad (20)$$

$$V(\varphi) = \frac{RF'(R) - F(R)}{2\kappa^2 F'^2(R)} \Big|_{R=R(\varphi)} = \frac{\alpha R^2 \exp(-\alpha R)}{2\kappa^2 (1 + \alpha R)^2} \Big|_{R=R(\varphi)}, \quad (21)$$

where the scalar curvature  $R$  in Eq.(21) being the solution of transcendental Eq.(20),  $R(\varphi)$ . It should be mentioned that in the Einstein frame free particles of matter do not move in space-time geodesics because of interactions with the scalar field  $\varphi$ . There is a correction in the right hand side of the geodesic equation representing a fifth force. Because the fifth force depends on space-time (and proportional to  $\nabla^\mu \varphi$ ) the universality of free fall (Weak Equivalence Principle) is violated. The mass squared of a scalar state is defined by the equation [3]

$$m_\varphi^2 = \frac{d^2 V}{d\varphi^2} = \frac{1}{3} \left( \frac{1}{F''(R)} + \frac{R}{F'(R)} - \frac{4F(R)}{F'^2(R)} \right). \quad (22)$$

We obtain from Eq.(1) the mass squared of a scalar field

$$m_\varphi^2 = \frac{[1 - 4\alpha R + (\alpha R)^3] \exp(-\alpha R)}{3\alpha (2 + \alpha R) (1 + \alpha R)^2}. \quad (23)$$

For  $R_0 = 0$  the value  $m_\varphi^2 = 1/(6\alpha)$  is positive and the solution with zero scalar curvature, corresponding to the Minkowski space, gives a stabile state.

The de Sitter solution (10)  $R_0 = 1/\alpha$  leads to the negative mass squared  $m_\varphi^2 = -1/(18e\alpha)$ . Thus, the de Sitter space with  $R_0 = 1/\alpha$  is unstable. For small value of  $\alpha$  corrections to the Newton law are negligible. One can say that after the Big Bang the Universe is in unstable de Sitter's phase and inflates (rapidly expands) having the positive curvature  $R_0 = 1/\alpha$ . Then the curvature decreases and the Universe approaches to the stable Minkowski space.

## 5 Matter Stability

From equation (6), after taking the trace, one obtains the equation of motion for a curvature scalar

$$3g^{\alpha\beta}\nabla_\alpha\nabla_\beta F'(R) + F'(R)R - 2F(R) = \kappa^2 T^{mat}, \quad (24)$$

where  $T^{mat} = T_{\mu\nu}^{mat} g^{\mu\nu}$ . Following [26], to investigate the matter stability, we consider Eq.(24) for weak gravity objects. For a flat Minkowski metric and spatially constant distribution Eq.(24) reads

$$-3F^{(2)}(R)\ddot{R} - 3F^{(3)}(R)\dot{R}^2 + F^{(1)}(R)R - 2F(R) = \kappa^2 T^{mat}, \quad (25)$$

with the notation  $F^{(n)}(R) = d^n F(R)/dR^n$ . From Eq.(1), we find

$$F^{(n)}(R) = \alpha^{n-1} (n + \alpha R) \exp(\alpha R). \quad (26)$$

Let us consider a perturbation so that  $R = R_0 + R_1$ ,  $|R_1| \ll |R_0|$ , and according to GR  $R_0 = -\kappa^2 T^{mat}$ . Then Eq.(25) leads to [5]

$$\begin{aligned} \ddot{R}_0 + \ddot{R}_1 + \frac{F^{(3)}(R_0)}{F^{(2)}(R_0)} (\dot{R}_0^2 + 2\dot{R}_0\dot{R}_1) \\ + \frac{2F(R_0) - R_0 [1 + F^{(1)}(R_0)]}{3F^{(2)}(R_0)} = U(R_0)R_1, \end{aligned} \quad (27)$$

with

$$\begin{aligned} U(R_0) = \frac{F^{(3)2} - F^{(2)}F^{(4)}}{F^{(2)2}} \dot{R}_0^2 \\ + \frac{(R_0 F^{(2)} - F^{(1)}) F^{(2)} + (2F - R_0 F^{(1)} - R_0) F^{(3)}}{3F^{(2)2}}. \end{aligned} \quad (28)$$

For  $U(R_0) > 0$ ,  $R_1$  exponentially increases in the time and the system is unstable. From Eq.(26), we obtain

$$U(R_0) = \frac{(\alpha \dot{R}_0)^2}{(2 + \alpha R_0)^2} + \frac{\alpha R_0 - 2}{3\alpha (2 + \alpha R_0)^2}. \quad (29)$$

Implying that the rate  $(\alpha \dot{R}_0)^2$  is small compared to  $1/\alpha$ ,  $(\alpha \dot{R}_0)^2 \ll 1/\alpha$ , for a matter stability  $U(R_0) < 0$ , one comes to the condition

$$R_0 < \frac{2}{\alpha}. \quad (30)$$

For static solutions (10) the inequality (30) is satisfied and the model passes the matter stability test. Eq.(30) gives the restriction on the biggest curvature of the Universe. One can introduce the fundamental length  $L = \sqrt{\alpha}$  so that the smallest size of the Universe is  $L$ . It should be mentioned that the Born–Infeld-like modified gravity model [27] possesses the similar features.

## 6 Conclusion

A novel modified theory of gravity with exponential-like Lagrangian density and the fundamental length  $L = \sqrt{\alpha}$  is suggested. We have considered  $F(R)$ -model that admits de Sitter solutions without a cosmological constant. Therefore, possibly the cosmic acceleration arises from new theory of gravity and GR is only an approximation describing the Universe at the intermediate cosmic time. At  $\alpha \rightarrow 0$  the action (2) approaches to Einstein–Hilbert action. From the bound (4) obtained, we find the restriction on the fundamental length  $L \leq 10^{-3}$  cm. We have found the static Schwarzschild–de Sitter solutions of the model and the potential of the scalar field in the scalar-tensor form of the theory. It was demonstrated that, for static solutions obtained, the de Sitter space is unstable and the Minkowski space with zero Ricci scalar is stable. Vacuum solutions are important in investigations of early and late time Universe. The model under consideration passes the matter stability test at  $R_0 < 2/\alpha$ . One can interpret this as a restriction on the size of the Universe at the beginning (the size of the Universe was greater than  $L$ ). To describe precisely the cosmological scenario, one needs to obtain exact solutions of equations of motion depending on time. This and other problems we leave for further investigations. We intend to verify the model introduced to be viable and an alternative to GR.



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